VI. *Project for a new Division of the Quadrant.* *By* Charles Hutton, *LL.D. F. R. S.* *In a Letter to the Rev. Dr.* Maſkelyne, *F. R. S. and Aſtronomer Royal.*

Read November 27, 1783.

Royal Mil. Acad. Woolwich,
Aug. 12, 1782.

DEAR SIR,

HAVING long ſince thought it would be a meritorious and uſeful ſervice, to adapt the tables of ſines, tangents, and ſecants, to equal parts of the radius inſtead of to thoſe of the quadrant; and having frequently mentioned this project to you, SIR, as a proper judge and promoter of all uſeful improvements in ſcience; I now beg leave to lay before you ſome obſervations I have thrown together on the ſubject, with a view to ſtimulate others, either to undertake and calculate ſome part of ſo large and painful a work, or to communicate farther hints for the improvement and eaſier performance of it.

I have the honour to be, &c.

A project

A project for constructing sines, tangents, secants, &c. to equal parts of the radius.

1. The arbitrary division of the quadrant of the circle into equal parts by 60ths, which has been delivered down to us from the ancients, and gradually extended by similar sub-divisions by the moderns, among various uses, serves for trigonometrical and other mathematical operations, by adapting to those divisions of the arc, certain lines expressed in equal parts of the radius, as chords, sines, tangents, &c. But among all the improvements in this useful branch of science, I have long wished to see a set of tables of sines, tangents, secants, &c. constructed to the arcs of the quadrant as divided into the like equal parts of the radius as those lines themselves. In this natural way, the arcs would not be expressed by divisions of 60ths, in degrees, minutes, &c., but by the common decimal scale of numbers; and the real lengths of the arcs, expressed in such common numbers, would then stand opposite their respective sines, tangents, &c. The uses of such an alteration would be many and great, and are too obvious and important to need pointing out or enforcing. I have therefore had for a long time a great desire to commence this arduous task; but continual interruptions have hitherto prevented me from making any considerable progress in so desirable an undertaking. But I am not without hopes that some future occasion may prove more propitious to my ardent wishes. It is not, however, to be expected, that this work can be accomplished by the labours of one person only; it will require rather the united endeavours of many. I shall therefore explain a few particulars relative to my project of this work, with a view to obtain from others, who

who may have leisure and abilities for it, their kind assistance, either by communicating hints of improvements, or by undertaking some part of the computations, to which they may be excited by their zeal for the accomplishment of so important a work, and by the extreme facility with which the calculations in this way are made.

2. In the first place then I would observe, that I think it will be sufficient to print the sines, tangents, &c. to seven places of figures; and that therefore it will be necessary to compute them to ten places, in order effectually to secure the truth of the seventh place to the nearest unit.

3. I would assume the radius equal to 100000, or suppose it to be divided into 100000 equal parts. Then it is well known, that the semi-circumference will be 314159·26536 nearly, and consequently the quadrant nearly 157079·63268 of the same equal parts, which is less than 157080 by ·36732, or nearly $\frac{1}{3}$ of an unit, or nearer $\frac{3}{8}=$·375, or nearer $\frac{4}{11}=$ ·3636, or still nearer $\frac{7}{19}=$·3684, or still nearer $\frac{11}{30}=$·36666 &c. And the half quadrant, or $\frac{1}{8}$ of the circle, 78539·81634 which is less than 78540 by only ·18366, or nearly $\frac{1}{2}$ only of any of the above-mentioned fractions.

4. The table may consist of five or more columns; the first column to contain the regular arithmetical series of arcs differing by unity, from the beginning, in this manner, 1, 2, 3, 4, 5, &c. up to half the quadrant, the next less whole number being 78539; then for the higher numbers, or those in the latter half quadrant, besides adding 1 continually, there must be at the first added the decimal ·63268, which will make all the numbers in this half become the exact complements of the first half, which consists of whole numbers only; and these will be the lengths of the arcs. Or, in order to include the

quadrantal arc 78539·81634, the firſt column may be continued up to 78540. The ſecond column to contain the correſponding degrees, minutes and ſeconds to the neareſt ſecond, or to the true ſeconds and decimals of a ſecond, for the convenience of eaſily changing the tables from the one meaſure to the other, or to make them anſwer to both methods; and the 3d, 4th, 5th, &c. columns to contain the correſponding ſines, tangents, ſecants, &c.

5. The tables may be diſpoſed as at preſent, namely, continuing them downwards by the left-hand ſide of the pages, as far as to the middle of the quadrant, and then returning them again backwards and upwards by the right-hand ſide of the pages.

6. In this diſpoſition, the numbers on the ſame line, the one on the left and the other on the right, will be exact complements of each other to a quadrant, and the decimal ·63268, in every number in the latter half quadrant, in each page, namely, either at the bottom of the column, or length-ways on the ſides of it.

7. A specimen of the first page of the table will therefore be this:

0									
Arcs	° ′ ″	Sines.	Tang.	Secants.	Cosec.	Cotan.	Cosin.	° ′ ″	Arcs
0	0 0 0	00000·00	00000·00	100000·00	infin.	infin.	100000·00	90 0 0	79
1	2	1·00							78
2	4	2·00							77
3	6	3·00							76
4	8	4·00							75
&c.									&c.
.									
.									
.									
.									
80									
.									
.									
.									
.									
&c.									&c.
96									83
97									82
98									81
99									80
Arcs	° ′ ″	Cosin.	Cotan.	Cosec.	Secants.	Tang.	Sines.	° ′ ″	Arcs

+·632679 &c.

8. To fill up the second column. Since the length of the quadrant is 157079·63267948966, and the number of seconds in the quadrant is 90 × 60 × 60 = 324000; therefore, as 157079·632 &c. : 324000 :: 1 : 2·062648062470964 = the number of seconds answering to each unit in our division of the quadrant, and which therefore being continually added will fill up the second column.

9. The number of seconds to be continually added being 2, and the decimal ·0626480624709 64, which is nearly equal to $\frac{1}{16}$, for $\frac{1}{16}$ is ·0625; therefore, besides adding 2 every time, we

we muſt alſo add 1″ more at every 16, which will make 3″ to be added at every 16th time, and 2″ at every other time beſides; but the firſt time the 3″ muſt be added will be at the arc or number 8, to have them to the neareſt ſecond, the repetition of the fraction at the arc 8 amounting to above $\frac{1}{2}$ a ſecond; and then the 3″ muſt be added at every 16 afterwards, *viz.* at 24, 40, 56, 72, 88, 104, &c.

10. But beſides the conſtant addition of 2″ every time, and of 1″ more every 16th time, there muſt be 1″ more added for every $6753\frac{1}{2}$ time, on account of the exceſs of the fraction ·062648062470964 over the fraction ·0625 or $\frac{1}{16}$: for that exceſs is ·000148062470964 which is $= \frac{1}{6753\frac{1}{2}}$. And the eaſieſt method of making this laſt addition of 1 at every $6753\frac{1}{2}$, will be to make the increaſe of the 1 on account of the $\frac{1}{16}$ at an unit ſooner for every $422\frac{3}{32}$; becauſe 16 is $422\frac{3}{32}$ times contained in $6753\frac{1}{2}$; by which means the incremental units for the $\frac{1}{16}$ will become 1 more at that number $6753\frac{1}{2}$, which laſt unit may be conſidered as the increment of the former increment for the $\frac{1}{16}$, and ſo proceed up to the quadrant; which will complete the ſecond column of arcs to the neareſt ſecond in each number. Or this ſecond column may be exactly computed to as many decimals as we pleaſe, by adding continually the 2″ and decimals, *viz.* 2·062648062470964. But at the middle of the quadrant, where the numbers return again upwards by the right-hand, there will for once be to be added only the ſeconds and decimals anſwering to the arc ·63268, *viz.* 1·30499618 ſeconds, that number being neceſſary to make the numbers on the right-hand to be the exact complements of thoſe on the left. Or it will, perhaps, be proper to make them to the neareſt unit in the 6th place of decimals. And to

fill up the ſecond column to this degree of accuracy, add continually 2·062648 ſeconds, but at the 9th line add 1 more, or 2·062649, becauſe 9 times 062470964 amounts to 56223867, or more than half a unit at that place; and after that add 1 more than 2·062648 at every 16th line, *viz.* at 25, 41, 57, 73, &c. becauſe 16 times 062470964 amounts to ·99953542, or nearly 1, it being only ·00046458 leſs than 1. And this number ·00046458, thus added too much, will, in 134 times adding it, amount to more than 06223867, the exceſs of 56223867 above 5, or half a unit, at that place; therefore at the line or number 2153 (or $9+16\times134$) which would be to have the 1 more added, let the 1 be there omitted, and add it at the next line or 2154, the true decimals after the firſt ſix, for 2153 being 499985, and for 2154 they are 562456. Continue thus always adding 1 more at every 16th line, except at the following numbers, where the 1 muſt be omitted, and added at the next following number; *viz.*

2153	10765	19377	27989	36601	45213	53825	62437	71033
4314	12926	21538	30150	38746	47358	55970	64582	73194
6459	15071	23683	32295	40907	49519	58131	66743	75339
8620	17232	25844	34456	43052	51664	60276	68888	77500

And thus proceed to the middle of the quadrant; by which means all the numbers will be to the neareſt unit in the ſixth or laſt place. Alſo, to have a check upon theſe numbers at certain intervals, it may be proper to proceed in this manner: Firſt find every 100th number, by adding its decimal ·264806 &c. verifying them at every 10th; then find every 16th number, by adding continually ·002369 &c. which will alſo be checked and verified at every 25th addition by one of the former ſet of 100, for 25 times 16 make 400, uſing a proper pre-

 caution

caution to preſerve each number true to the neareſt unit in the 6th or laſt decimal.

As to the decimals of the numbers in the latter half of the quadrant, they will be the complements, to 1, of the correſponding numbers in the firſt half; and therefore they may be all eaſily found by taking each figure from 9, and the laſt from 10. But it will be ſafeſt to find only every 10th decimal in this way, and to fill up the intermediate nine by adding, as before, the conſtant decimal 062648; by which means they will be checked and verified at every 10th number.

11. To fill up the third column, or that of ſines, as well as thoſe of tangents and ſecants, it may firſt be obſerved, that the old tables of thoſe lines to every minute, or even to every ten ſeconds of the quadrant, cannot be of ſo much uſe as it might ſeem at firſt ſight; as the very near coincidence of the numbers in the new and old diviſions appear very ſeldom to happen. I find, indeed, that our arc 1309 anſwers nearly to 45 minutes, that arc exceeding 45′ by only ·00632363 or $\frac{1}{158}$ part of a ſecond nearly, and ſo in proportion for their equi-multiples. But although this degree of coincidence may be ſufficient for checking the correſponding values of the arcs in the firſt and ſecond columns, we are not thereby authoriſed to conſider the ſine, tangent, or ſecant of 1309 as accurately equal to that of 45′ in all the ſeven places of figures, but differing from it by nearly the $\frac{1}{158}$ part of the difference correſponding to 1″, which is about $\frac{1}{3}$ of an unit in the ſines and tangents, though next to nothing in the ſecants. This, therefore, although it makes no ſenſible difference in this particular caſe, will cauſe a difference that muſt not be neglected in the equi-multiples of 1309 and 45′, the ſines and tangents of which will differ by half a unit or

 more,

more, and therefore will not be expreſſed by the ſame number, but will have ſome ſmall difference in the ſeventh or laſt figure. And the ſame will happen in almoſt all the other arcs; ſo that generally the ſines, &c. which are exact for the arcs in the firſt column, will not be quite ſo for thoſe in the ſecond, when expreſſed in whole ſeconds only, ſince theſe will ſometimes differ by the part correſponding to almoſt half a ſecond. However, in this, or any other caſe, where the difference is exactly known, we may profitably make uſe of the numbers in the old tables for conſtructing or verifying thoſe of the new, by taking in the proportional part of the difference. Let, therefore, all the ſines, &c. of every 1309 be computed from the old tables, and entered in the new, by adding to the ſine, &c. of the correſponding multiple of 45′ the like multiple of the $\frac{1}{138}$ part of the proportional difference for 1″. This will give about 120 ſines, &c. to ſerve as a verification of the computations by the more general methods. But if the ſecond column be exactly conſtructed with all its decimal places by the continual addition of 2·06264807, the old tables may be converted into the new, by allowing for the odd ſeconds and decimals. And for this purpoſe it will, perhaps, be beſt to uſe the large table of RHETICUS, which contains the ſines, tangents, and ſeconds, to ten places of figures for every 10″, and alſo the differences. At leaſt, ſuch ſines, &c. may be found in this way as have their ſeconds and decimals well adapted for the purpoſe; and for ſuch as would be found too troubleſome in this way, recourſe may be had to ſome of the following methods.

12. Let us now examine the expreſſions for the ſines, &c. by infinite ſeries.

The

The radius being 1, and arc a, it is well known that the

$$\text{sine is } = a - \tfrac{1}{6}a^3 + \tfrac{1}{120}a^5 - \tfrac{1}{5040}a^7 + \tfrac{1}{362880}a^9 - \tfrac{1}{39916800}a^{11} \ \&c.$$

$$\text{cosine } = 1 - \tfrac{1}{2}a^2 + \tfrac{1}{24}a^4 - \tfrac{1}{720}a^6 + \tfrac{1}{40320}a^8 - \tfrac{1}{3628800}a^{10} \ \&c.$$

$$\text{tang. } = a + \tfrac{1}{3}a^3 + \tfrac{2}{15}a^5 + \tfrac{17}{315}a^7 + \tfrac{62}{2835}a^9 + \tfrac{1382}{155925}a^{11} \ \&c.$$

$$\text{cotang.} = a^{-1} - \tfrac{1}{3}a - \tfrac{1}{45}a^3 - \tfrac{2}{945}a^5 - \tfrac{1}{4725}a^7 - \tfrac{2}{93555}a^9 \ \&c.$$

$$\text{secant } = 1 + \tfrac{1}{2}a^2 + \tfrac{5}{24}a^4 + \tfrac{61}{720}a^6 + \tfrac{277}{8064}a^8 + \tfrac{50521}{3628800}a^{10} \ \&c.$$

$$\text{cosec. } = a^{-1} + \tfrac{1}{6}a + \tfrac{7}{360}a^3 + \tfrac{31}{15120}a^5 + \tfrac{127}{604800}a^7 + \tfrac{73}{3421440}a^9 \ \&c.$$

Or the same series are thus otherwise expressed:

$$\text{sine } = a - \frac{1}{2.3}a^3 + \frac{b}{4.5}a^3 - \frac{c}{6.7}a^7 + \frac{d}{8.9}a^9 - \frac{e}{10.11}a^{11} \ \&c.$$

$$\text{cosine } = 1 - \frac{1}{2}a^2 + \frac{b}{3.4}a^4 - \frac{c}{5.6}a^6 + \frac{d}{7.8}a^8 - \frac{e}{9.10}a^{10} \ \&c.$$

$$\text{tangent} = a + \frac{1}{1.3}a^3 + \frac{8b}{4.5}a^5 + \frac{17c}{6.7}a^7 + \frac{29\frac{3}{17}d}{8.9}a^9 + \frac{44\frac{18}{31}e}{10.11}a^{11} \ \&c.$$

$$\text{cotang.} = a^{-1} - \frac{1}{3}a - \frac{b}{15}a^3 - \frac{2c}{21}a^5 - \frac{d}{10}a^7 - \frac{10e}{99}a^9 \ \&c.$$

$$\text{secant } = 1 + \frac{1}{2}a^2 + \frac{5b}{12}a^4 + \frac{61c}{150}a^6 + \frac{1385d}{3416}a^8 + \frac{50521e}{124651}a^{10} \ \&c.$$

$$\text{cosec. } = a^{-1} + \frac{1}{6}a + \frac{7}{60}a^3 + \frac{31c}{294}a^5 + \frac{127d}{1240}a^7 + \frac{2555e}{25146}a^9 \ \&c.$$

where b, c, d, e, &c. denote the preceding co-efficients. And hence, with the help of the table of the first ten powers of the first 100 numbers, in p. 101. of my tables of powers published by order of the Board of Longitude, may be easily found the sines, &c. of all arcs up to 100, by only dividing those powers by their respective co-efficients, as also of all multiples of these arcs by 10, 100, &c. by only varying the decimal points in the several terms, as the figures will be all the same: and

and thus a number of primary fines, &c. may be found, to check or verify the fame when computed by other methods. By this method will be found the fines, &c. of the arcs

1, 10, 100, 1000, 10000, 100000;
2, 20, 200, 2000, 20000;
3, 30, 300, 3000, 30000;
4, 40, 400, 4000, 40000;
&c. till
99, 990, 9900, 99000, 990000.

13. Again, it is evident, that, of the terms in the feries for the fine, the firft term a alone will give the fine true to the neareft unit in the ninth place in the firft 144 fines, or the arc and fine will be the fame for nine places as far as the arc 144; but they will agree to the neareft unit in the feventh place as far as the arc 669; after which the fecond term of the feries muft be included.

14. When the fecond term is taken in, thefe two terms $a - \frac{1}{6}a^3$ will give the fines true to the neareft unit in the ninth place till the arc becomes 3500. Now the numbers in my table of cubes (juft publifhed by order of the Board of Longitude) extend to 10000, and therefore all the above cubes are found in it; confequently taking the fixth part of thofe cubes, and fubtracting it from the correfponding arcs, the remainders will be the fines of thofe arcs, as far as till the arc be 3500: after which the third term of the feries may be taken in, or other methods may be ufed.

15. But fince, for any arc a, this is a general theorem, *viz.* as radius : 2 cof. a :: fin. na : fin. $\overline{n-1} \times a +$ fin. $\overline{n+1} \times a$; taking $a = 1$, radius 10000, the fine of a will be $1 - \cdot 0000000000\frac{1}{6}$ and the cofine of a will be $100000 - \cdot 000005$, and the above

above proportion will become 100000 : 200000 − ·00001, or 1 : 2 − ·0000000001 :: sin. n : sin. $\overline{n-1}$ + sin. $\overline{n+1}$; consequently sin. $\overline{n-1}$ + sin. $\overline{n+1}$ is = 2 sin. n. − 0000000001 sin. n, and the sines are in arithmetical progression except only for the small difference of ·0000000001 sin. n, hence sin. $\overline{n+1}$ is = $\overline{2-\cdot 0000000001}$ × sin. n − sin. $\overline{n-1}$; and therefore taking n successively equal to 1, 2, 3, 4, &c. the series of sines will be as follows:

sin. 1 = 1 − ·0000000000$\frac{1}{6}$;
sin. 2 = 2 − ·0000000001 × sin. 1;
sin. 3 = 2 − ·0000000001 × sin. 2 − sin. 1;
sin. 4 = 2 − ·0000000001 × sin. 3 − sin. 2;
sin. 5 = 2 − ·0000000001 × sin. 4 − sin. 3;
&c.

And by this theorem, *viz.* sin. $\overline{n+1}$ = $\overline{2-\cdot 0000000001}$ × sin. n − sin. $\overline{n-1}$, may be easily filled up the intervals between those primary numbers mentioned in former articles.

16. In like manner, as radius : 2 cos. a :: cos. na : cos. $\overline{n-1}\,.\,a$ + cos. $\overline{n+1}\,.\,a$; and hence this theorem, cos. $\overline{n+1}$ = $\overline{2\quad\cdot 0000000001}$ × cos. n − cos. $\overline{n-1}$, by which the cosines will be all easily filled up. And these two theorems for the sines and cosines are so easy and accurate, that we need not have recourse to any other, but only to check and verify these at certain intervals, as at every 100th number, by a proportion from RHETICUS's canon, as mentioned at art. 11. or by any other way.

17. The sines and cosines being compleated, the difference between the radius and cosine will be the versed sine; the difference between radius and sine will be the co-versed sine; and the sum of the radius and cosine will be the sup.versed sine.

18. From

18. From the ſines and coſines alſo, the tangents, cotangents, ſecants, and coſecants, may be made by theſe known proportions, *viz.* as

1. coſine : radius :: ſine : tangent,
2. ſine : radius :: coſine : cotangent,
3. coſine : radius :: radius : ſecant,
4. ſine : radius :: radius : coſecant,
5. radius : ſine :: ſecant : tangent,
6. radius : coſine :: coſecant : cotangent,
7. tangent : radius :: radius : cotangent.

Wherefore, the reciprocal of the coſine will be the ſecant; the reciprocal of the ſine, the coſecant; the quotient of the ſine by the coſine, the tangent; and the quotient of the coſine by the ſine, the cotangent; or the product of the ſine and ſecant will be the tangent, and the product of the coſine and coſecant, the cotangent; or, laſtly, the reciprocal of the tangent is the cotangent; proper regard being had to the number of decimals, on account of our radius being 100000 inſtead of 1 only.

And theſe are to be uſed when the application happens to be eaſier than the general ſeries, and eaſier than by proportion from RHETICUS's canon.

But there are other particular theorems, which, by a little addreſs, may be rendered more expeditious than any of the former: thus,

19. In any two arcs this is a general proportion,

As the difference of their ſines :
to the ſum of their ſines ::
ſo tangent of half the difference of the arcs :
to tangent of half their ſum.

So that by taking continually the arcs, having the common difference 2, the third term of this proportion will be 1, and the fourth term will be found by dividing the ſum of the ſines

by their difference, which divisor or difference will never consist of more than four or five figures, *viz.* about half the number of figures that are in the divisors mentioned in the preceding article.

20. Again, As the difference of the cosines :
to the sum of the cosines ::
so tangent of half their difference :
to tangent of half their sum.

And thus the cotangents will be found by dividing the sum of the cosines of two arcs, differing by 2, by their small difference.

21. Also the secant of an arc is equal to the sum of its tangent and the tangent of half its complement; and the cosecant of an arc is equal to the sum of its cotangent and the tangent of half the arc; or half the sum of the tangent and cotangent is equal to the cosecant of the double arc. From whence the secants and cosecants will be easily made.

22. Thus I have pointed out methods by which the whole tables may be readily constructed. Should any other useful methods or improvements occur to any person, the communication of them to me will be thankfully received. I am now engaged in making some of the computations; and it is hoped, that the facility of them, with the desireableness of the tables, will induce some ingenious lovers of the mathematics to lend their aid in performing some part of the work. Should any such be so inclined, before he begins, I must request he will be pleased to signify his intention to me, that I may point out to him such parts of the work as have not before been performed or undertaken, to prevent the chance of losing his labour by re-computing any parts that may have been already executed by myself or others.

Royal Military Academy,
Woolwich, Aug. 12, 1782.

CHARLES HUTTON.